Problem 1.62

The integral

$$\mathbf{a} \equiv \int_{\mathcal{S}} d\mathbf{a} \tag{1.106}$$

is sometimes called the **vector area** of the surface $\mathcal{S}$. If $\mathcal{S}$ happens to be *flat*, then $|\mathbf{a}|$ is the *ordinary* (scalar) area, obviously.

(a) Find the vector area of a hemispherical bowl of radius R.

(b) Show that $\mathbf{a} = \mathbf{0}$ for any *closed* surface. [*Hint:* Use Prob. 1.61a.]

(c) Show that $\mathbf{a}$ is the same for all surfaces sharing the same boundary.

(d) Show that

$$\mathbf{a} = \frac{1}{2} \oint \mathbf{r} \times d\mathbf{l}, \tag{1.107}$$

where the integral is around the boundary line. [*Hint:* One way to do it is to draw the cone subtended by the loop at the origin. Divide the conical surface up into infinitesimal triangular wedges, each with vertex at the origin and opposite side $d\mathbf{l}$, and exploit the geometrical interpretation of the cross product (Fig. 1.8).]

(e) Show that

$$\oint (\mathbf{c} \cdot \mathbf{r})\, d\mathbf{l} = \mathbf{a} \times \mathbf{c}, \tag{1.108}$$

for any constant vector $\mathbf{c}$. [*Hint:* Let $T = \mathbf{c} \cdot \mathbf{r}$ in Prob. 1.61e.]

Solution

Part (a)

The outward normal unit vector of a hemispherical bowl centered at the origin is $\hat{\mathbf{r}}$. Use spherical coordinates (r, ϕ, θ), where θ is the angle from the polar axis.

$$\begin{aligned}
\mathbf{a} \equiv \iint_S d\mathbf{S} &= \int_0^{\pi/2} \int_0^{2\pi} \hat{\mathbf{r}} R^2 \sin\theta \, d\phi \, d\theta \\
&= \int_0^{\pi/2} \int_0^{2\pi} (\sin\theta \cos\phi \, \hat{\mathbf{x}} + \sin\theta \sin\phi \, \hat{\mathbf{y}} + \cos\theta \, \hat{\mathbf{z}}) R^2 \sin\theta \, d\phi \, d\theta \\
&= R^2 \left[\int_0^{\pi/2} \int_0^{2\pi} (\hat{\mathbf{x}} \sin^2\theta \cos\phi + \hat{\mathbf{y}} \sin^2\theta \sin\phi + \hat{\mathbf{z}} \sin\theta \cos\theta) \, d\phi \, d\theta \right] \\
&= R^2 \left(\hat{\mathbf{x}} \int_0^{\pi/2} \int_0^{2\pi} \sin^2\theta \cos\phi \, d\phi \, d\theta + \hat{\mathbf{y}} \int_0^{\pi/2} \int_0^{2\pi} \sin^2\theta \sin\phi \, d\phi \, d\theta \right. \\
&\qquad \left. + \hat{\mathbf{z}} \int_0^{\pi/2} \int_0^{2\pi} \sin\theta \cos\theta \, d\phi \, d\theta \right)
\end{aligned}$$

Evaluate each of the double integrals.

$$\begin{aligned}\mathbf{a} = R^2 \Bigg[\hat{\mathbf{x}} \left(\int_0^{\pi/2} \sin^2\theta \, d\theta \right) \left(\int_0^{2\pi} \cos\phi \, d\phi \right) \\ + \hat{\mathbf{y}} \left(\int_0^{\pi/2} \sin^2\theta \, d\theta \right) \left(\int_0^{2\pi} \sin\phi \, d\phi \right) \\ + \hat{\mathbf{z}} \left(\int_0^{\pi/2} \sin\theta\cos\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right) \Bigg] \\ = R^2 \left[\hat{\mathbf{x}} \left(\int_0^{\pi/2} \sin^2\theta \, d\theta \right)(0) + \hat{\mathbf{y}} \left(\int_0^{\pi/2} \sin^2\theta \, d\theta \right)(0) + \hat{\mathbf{z}} \left(\frac{1}{2} \right)(2\pi) \right] \\ = \pi R^2 \hat{\mathbf{z}}\end{aligned}$$

Part (b)

Use the result of part (a) from Problem 1.61.

$$\iiint_D \nabla T \, dV = \oiint_{\text{bdy } D} T \, d\mathbf{S}$$

Set $T = 1$.

$$\begin{aligned}\oiint_{\text{bdy } D} d\mathbf{S} &= \iiint_D (\nabla 1) \, dV \\ &= \iiint_D \left\langle \frac{\partial}{\partial x}(1), \frac{\partial}{\partial y}(1), \frac{\partial}{\partial z}(1) \right\rangle dV \\ &= \iiint_D \langle 0, 0, 0 \rangle \, dV \\ &= \iiint_D (\mathbf{0}) \, dV \\ &= \mathbf{0}\end{aligned}$$

Part (c)

Use the result of part (d).

$$\mathbf{a} = \iint_S d\mathbf{S} = \frac{1}{2} \oint_{\text{bdy } S} \mathbf{r} \times d\mathbf{l}$$

It says that regardless of what surface S is being integrated over, $\mathbf{a}$ is only dependent on the surface's boundary line, bdy S.

Part (d)

The aim here is to show that

$$\iint_S d\mathbf{S} = \frac{1}{2}\oint_{\text{bdy } S} \mathbf{r}\times d\mathbf{l},$$

where S is an open surface and bdy S is this surface's boundary line. We want to use Stokes's theorem on the right side, but there's a cross product in the integrand rather than a dot product. So consider the dot product of $\mathbf{c}$, an arbitrary constant vector, with this integral.

$$\mathbf{c}\cdot\frac{1}{2}\oint_{\text{bdy } S} \mathbf{r}\times d\mathbf{l} = \frac{1}{2}\oint_{\text{bdy } S} \mathbf{c}\cdot(\mathbf{r}\times d\mathbf{l})$$

Use Identity 1.

$$\mathbf{c}\cdot\frac{1}{2}\oint_{\text{bdy } S} \mathbf{r}\times d\mathbf{l} = \frac{1}{2}\oint_{\text{bdy } S} (\mathbf{c}\times\mathbf{r})\cdot d\mathbf{l}$$

Now apply Stokes's theorem to turn this closed loop integral into a surface integral.

$$\begin{aligned}
\mathbf{c}\cdot\frac{1}{2}\oint_{\text{bdy } S} \mathbf{r}\times d\mathbf{l} &= \frac{1}{2}\iint_S [\nabla\times(\mathbf{c}\times\mathbf{r})]\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\left(\sum_{j=1}^{3}\boldsymbol{\delta}_j c_j\right)\times\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k x_k\right)\right]\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\sum_{j=1}^{3}\sum_{k=1}^{3}(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)c_j x_k\right]\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left(\sum_{i=1}^{3}\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\boldsymbol{\delta}_l\varepsilon_{jkl}c_j x_k\right)\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}\frac{\partial}{\partial x_i}c_j x_k\right]\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{ilm}\varepsilon_{jkl}\left(\frac{\partial c_j}{\partial x_i}x_k + c_j\frac{\partial x_k}{\partial x_i}\right)\right]\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}\left[(0)x_k + c_j(\delta_{ik})\right]\right]\cdot d\mathbf{S} \\
&= \frac{1}{2}\iint_S \left[\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})c_j\delta_{ik}\right]\cdot d\mathbf{S}
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\mathbf{c}\cdot\frac{1}{2}\oint_{\text{bdy } S}\mathbf{r}\times d\mathbf{l} &= \frac{1}{2}\iint_S\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m(\delta_{mj}\delta_{kk}-\delta_{mk}\delta_{kj})c_j\right]\cdot d\mathbf{S}\\
&= \frac{1}{2}\iint_S\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{kk}c_j-\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{kj}c_j\right)\cdot d\mathbf{S}\\
&= \frac{1}{2}\iint_S\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_j\delta_{kk}c_j-\sum_{j=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\delta_{kj}c_j\right)\cdot d\mathbf{S}\\
&= \frac{1}{2}\iint_S\left[\left(\sum_{k=1}^{3}\delta_{kk}\right)\sum_{j=1}^{3}\boldsymbol{\delta}_jc_j-\sum_{j=1}^{3}\boldsymbol{\delta}_jc_j\right]\cdot d\mathbf{S}\\
&= \frac{1}{2}\iint_S\left(3\sum_{j=1}^{3}\boldsymbol{\delta}_jc_j-\sum_{j=1}^{3}\boldsymbol{\delta}_jc_j\right)\cdot d\mathbf{S}\\
&= \frac{1}{2}\iint_S\left(2\sum_{j=1}^{3}\boldsymbol{\delta}_jc_j\right)\cdot d\mathbf{S}\\
&= \iint_S\left(\sum_{j=1}^{3}\boldsymbol{\delta}_jc_j\right)\cdot d\mathbf{S}\\
&= \iint_S\mathbf{c}\cdot d\mathbf{S}\\
&= \mathbf{c}\cdot\iint_S d\mathbf{S}
\end{aligned}$$

Therefore,

$$\iint_S d\mathbf{S} = \frac{1}{2}\oint_{\text{bdy } S}\mathbf{r}\times d\mathbf{l}.$$

Part (e)

Use the result of part (e) from Problem 1.61.

$$\iint_S \nabla T\times d\mathbf{S} = -\oint_{\text{bdy } S} T\,d\mathbf{l}$$

Set $T=\mathbf{c}\cdot\mathbf{r}$, where $\mathbf{c}$ is a constant vector and $\mathbf{r}$ is the position vector.

$$\iint_S \nabla(\mathbf{c}\cdot\mathbf{r})\times d\mathbf{S} = -\oint_{\text{bdy } S}(\mathbf{c}\cdot\mathbf{r})\,d\mathbf{l}$$

Solve for the line integral and simplify.

$$
\begin{aligned}
\oint_{\text{bdy } S} (\mathbf{c}\cdot\mathbf{r})\, d\mathbf{l} &= -\iint_S \nabla(\mathbf{c}\cdot\mathbf{r}) \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j c_j\right) \cdot \left(\sum_{k=1}^{3} \boldsymbol{\delta}_k x_k\right)\right] \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \left[\sum_{j=1}^{3}\sum_{k=1}^{3} (\boldsymbol{\delta}_j \cdot \boldsymbol{\delta}_k) c_j x_k\right] \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \left(\sum_{j=1}^{3}\sum_{k=1}^{3} \delta_{jk} c_j x_k\right) \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \left(\sum_{j=1}^{3} c_j x_j\right) \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i} c_j x_j\right) \times d\mathbf{S} \\
&= -\iint_S \left[\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \left(\frac{\partial c_j}{\partial x_i} x_j + c_j \frac{\partial x_j}{\partial x_i}\right)\right] \times d\mathbf{S} \\
&= -\iint_S \left\{\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i \left[(0)x_j + c_j(\delta_{ij})\right]\right\} \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{\delta}_i c_j \delta_{ij}\right) \times d\mathbf{S} \\
&= -\iint_S \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i c_i\right) \times d\mathbf{S} \\
&= -\iint_S \mathbf{c} \times d\mathbf{S} \\
&= -\mathbf{c} \times \iint_S d\mathbf{S} \\
&= -\mathbf{c} \times \mathbf{a} \\
&= \mathbf{a} \times \mathbf{c}
\end{aligned}
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